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Extremal digraphs containing at most t paths of length 2 with the same endpoints.

All digraphs considered in this paper are simple (that is, without loops or parallel arcs). For a positive integer t, denote by $P_{t,2}$ the orientation of the complete bipartite graph $K_{2,t}$ consisting of t directed paths of length 2 sharing the same initial and terminal vertices. Let $\mathrm{ex}(n, P_{t,2})$ be the maximum size of a $P_{t,2}$-free digraph of order n. The problem of determining $\mathrm{ex}(n, P_{t,2})$ was resolved by the same authors for $t = 2$ in [Discrete Math. 343, No. 5, Article ID 111827, 16 p. (2020; Zbl 1435.05093)] for all $n \geq 13$, along with a complete characterization of the extremal graphs.

Define $g(n,t) = \lceil (n+t)/2 \rceil \cdot \lfloor (n-t)/2 \rfloor + tn + 1$. In this paper, the authors show that for $t \geq 2$ and $n \geq \max\{t^3 + 4t^2 + 3t + 4,\ 17t^2/2 + 30t + 27\}$, we have $\mathrm{ex}(n, P_{t+1,2}) = g(n,t)$ if $\lfloor (n-t)/2 \rfloor$ is odd, and $\mathrm{ex}(n, P_{t+1,2}) \in \{g(n,t) - 1, g(n,t)\}$ if $\lfloor (n-t)/2 \rfloor$ is even. It is an open problem to precisely determine $\mathrm{ex}(n, P_{t+1,2})$ in the latter case, and the authors conjecture that the right answer is $g(n,t) - 1$. A complete characterization of the extremal graphs also remains open.

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